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Schwinger Formula Revisited

Jaume Haro[1]

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We apply the formal W.K.B. method in the complex plane to the quantum field theory to obtain the Schwinger formula for spin and spinless particles; i.e., we obtain the probability that the vacuum state remains unchanged in presence of a constant electric field. Finally, from Schwinger formula we calculate the probability that *n* pairs are produced.

KEY WORDS: pair production; Schwinger's formula; semiclassical approach.

1. INTRODUCTION

In this work we deduce the well-known Schwinger formula for spin and spinless particles, using the W.K.B. approximation, i.e., we compute the probability that the vacuum state remains unchanged in the presence of a constant electric field, using the semiclassical approach. Once we have deduced the Schwinger formula, we obtain the probability that *n* pairs are produced. This is the main result of the paper.

In the first section, we study the Klein–Gordon field coupled with a uniform electric field. It is a well-known fact that the Klein-Gordon field is equivalent to a Hamiltonian system composed of an infinite number of harmonic oscillators with frequencies which depend on time. Then, using the Bogolubov transformation, we see that the probability that no pairs in the $\vec{k}$-state are produced is equal to the transmission coefficient, and the relative pair production probability is equal to the corresponding reflection coefficient. We also deduce that the average number of produced pairs is the penetration factor.

To obtain the Schwinger formula for boson particles, we apply the previous results to the case of a constant electric field. The Schwinger formula for fermion particles is obtained using the Exclusion Principle.

Finally, using the previous results, we deduce the general formula that gives the probability that *n* pairs are produced in the presence of a constant electric field.

[1] Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain; e-mail: jaime.haro@upc.es.

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2. THE KLEIN–GORDON FIELD COUPLED WITH A VECTOR POTENTIAL

We consider the Klein-Gordon field in a box of volume L^3, coupled with an external uniform vector potential $\vec{f}(t)$. Then the Klein-Gordon equation is equivalent to a Hamiltonian system (Bjorken and Drell, 1965; Haro, in press), composed of an infinite number of harmonic oscillators with frequencies which depend on time. The Hamilton equations are

$$u'' + \omega_{\vec{k}}^2(t)u = 0, \qquad \forall \vec{k} \in \mathbb{Z}^3, \tag{1}$$

where

$$\omega_{\vec{k}}^2(t) = \frac{1}{\hbar^2}\left(c^2\left|\frac{2\pi\hbar\vec{k}}{L} + \frac{e}{c}\vec{f}(t)\right|^2 + m^2c^4\right).$$

We suppose that $\lim_{t\to\pm\infty}\omega_{\vec{k}}(t) = \omega_{\vec{k}}^{\pm}$. Then, Eq. (1) has the two following fundamental systems of solutions (Fedoryuk, 1993):

1. The system "in," composed of the functions $u_{\mathrm{in},\vec{k}}^{\pm}(t)$ that have the asymptotic behavior
$$\lim_{t\to-\infty}\left[u_{\mathrm{in},\vec{k}}^{\pm}(t) - \sqrt{\frac{1}{2\hbar\omega_{\vec{k}}^{-}}}\exp(\pm i\omega_{\vec{k}}^{-}t)\right] = 0.$$
2. The system "out," composed of the functions $u_{\mathrm{out},\vec{k}}^{\pm}(t)$ that have the asymptotic behavior
$$\lim_{t\to+\infty}\left[u_{\mathrm{out},\vec{k}}^{\pm}(t) - \sqrt{\frac{1}{2\hbar\omega_{\vec{k}}^{+}}}\exp(\pm i\omega_{\vec{k}}^{+}t)\right] = 0.$$

Now we write the "in"-states as linear combinations of the "out"-states

$$u_{\mathrm{in},\vec{k}}^{-}(t) = \frac{1}{T_{\vec{k}}}u_{\mathrm{out},\vec{k}}^{-}(t) + \frac{R_{\vec{k}}}{T_{\vec{k}}}u_{\mathrm{out},\vec{k}}^{+}(t); \quad u_{\mathrm{in},\vec{k}}^{+}(t) = \frac{1}{T_{\vec{k}}^{*}}u_{\mathrm{out},\vec{k}}^{+}(t) + \frac{R_{\vec{k}}^{*}}{T_{\vec{k}}^{*}}u_{\mathrm{out},\vec{k}}^{-}(t),$$

where we have used that

$$u^{+}_{\binom{\mathrm{in}}{\mathrm{out}}} = \left(u^{-}_{\binom{\mathrm{in}}{\mathrm{out}}}\right)^{*}.$$

Therefore, for the field $\hat{\psi}_{\vec{k}} \equiv \hat{a}_{\mathrm{in},\vec{k}}u_{\mathrm{in},\vec{k}}^{-} + \hat{b}_{\mathrm{in},\vec{k}}^{+}u_{\mathrm{in},\vec{k}}^{+}$ we have

$$\hat{\psi}_{\vec{k}} = \left(\hat{a}_{\mathrm{in},\vec{k}}\frac{1}{T_{\vec{k}}} + \hat{b}_{\mathrm{in},\vec{k}}^{+}\frac{R_{\vec{k}}^{*}}{T_{\vec{k}}^{*}}\right)u_{\mathrm{out},\vec{k}}^{-} + \left(\hat{a}_{\mathrm{in},\vec{k}}\frac{R_{\vec{k}}}{T_{\vec{k}}} + \hat{b}_{\mathrm{in},\vec{k}}^{+}\frac{1}{T_{\vec{k}}^{*}}\right)u_{\mathrm{out},\vec{k}}^{+}.$$

From this expression, we obtain the "out" operators (Fulling, 1985),

$$\hat{a}_{\text{out},\vec{k}} = \hat{a}_{\text{in},\vec{k}}\frac{1}{T_{\vec{k}}} + \hat{b}^{+}_{\text{in},\vec{k}}\frac{R^*_{\vec{k}}}{T^*_{\vec{k}}}, \quad \hat{b}^{+}_{\text{out},\vec{k}} = \hat{a}_{\text{in},\vec{k}}\frac{R_{\vec{k}}}{T_{\vec{k}}} + \hat{b}^{+}_{\text{in},\vec{k}}\frac{1}{T^*_{\vec{k}}}.$$

Let $|n_{\vec{k}}\rangle$ be the "in"-state that contains n pairs, and let $|n_{\vec{k}})$ be the "out"-state that contains n pairs. Then, it is easy to obtain the following relations (Grib *et al.*, 1994):

$$|0_{\vec{k}}\rangle = \tilde{C}_{\vec{k}}\sum_{n=0}^{\infty}(R^*_{\vec{k}})^n|n_{\vec{k}}), \quad |0_{\vec{k}}) = C_{\vec{k}}\sum_{n=0}^{\infty}\left(-\frac{T_{\vec{k}}}{T^*_{\vec{k}}}R^*_{\vec{k}}\right)^n|n_{\vec{k}}\rangle,$$

with $|\tilde{C}_{\vec{k}}|^2 = |C_{\vec{k}}|^2 = |T_{\vec{k}}|^2$.

From these relations, we deduce that the probability that a pair in the $\vec{k}$-state is produced (Gribb *et al.*, 1994; Marinov and Popov, 1977; Parker, 1969) is

$$|(n_{\vec{k}}|0_{\vec{k}}\rangle|^2 = |T_{\vec{k}}|^2|R_{\vec{k}}|^{2n},$$

and the average number of produced pairs in the $\vec{k}$-state is

$$\langle 0_{\vec{k}}|\hat{a}^{+}_{\text{out},\vec{k}}\hat{a}_{\text{out},\vec{k}}|0_{\vec{k}}\rangle = \langle 0_{\vec{k}}|\hat{b}^{+}_{\text{out},\vec{k}}\hat{b}_{\text{out},\vec{k}}|0_{\vec{k}}\rangle = \frac{|R_{\vec{k}}|^2}{|T_{\vec{k}}|^2}.$$

Now, if we use the penetration factor, $p_{\vec{k}} \equiv \frac{|R_{\vec{k}}|^2}{|T_{\vec{k}}|^2}$, from the equations

$$|T_{\vec{k}}|^2\sum_{n=0}^{\infty}|R_{\vec{k}}|^{2n} = 1 \quad \text{(total probability equal to 1)},$$

$$|T_{\vec{k}}|^2\sum_{n=0}^{\infty}n|R_{\vec{k}}|^{2n} = p_{\vec{k}} \quad \text{(average number of produced pairs in the } \vec{k}\text{-state)},$$

we obtain the transmissions and reflection coefficients $|T_{\vec{k}}|^2 = \frac{1}{1+p_{\vec{k}}}$, $|R_{\vec{k}}|^2 = \frac{p_{\vec{k}}}{1+p_{\vec{k}}}$.

2.1. Pair Production

Let P_n be the probability that n pairs are produced, then

$$P_n = \sum_{\vec{j}1\geq\cdots\geq\vec{j}n}\prod_{s=1}^{n}|R_{\vec{j}s}|^2\prod_{\vec{j}\in\mathbb{Z}^3}|T_{\vec{j}}|^2.$$

To obtain a simple expression of P_n we use the following generating function $g(x) = \prod_{\vec{j}\in\mathbb{Z}^3}\frac{1}{1-x|R_{\vec{j}}|^2}$ (see Nikishov, 1970). It is easy to verify that

$$g(x) = 1 + \sum_{n=1}^{\infty}x^n\sum_{\vec{j}1\geq\cdots\geq\vec{j}n}\prod_{s=1}^{n}|R_{\vec{j}s}|^2,$$

consequently, we have $D^n g(0) = n! \sum_{\vec{j}1 \geq \cdots \geq \vec{j}n} \prod_{s=1}^{n} |R_{\vec{j}s}|^2$. On the other hand $g(1) = \prod_{\vec{j} \in \mathbb{Z}^3} \frac{1}{|T_{\vec{j}}|^2} = \frac{1}{P_0}$, therefore, we obtain

$$P_n = \frac{1}{n!} \frac{D^n g(0)}{g(1)}. \tag{2}$$

Now using the penetration factor, we obtain the formulae:

$$P_0 = \prod_{\vec{k} \in \mathbb{Z}^3} \frac{1}{1 + p_{\vec{k}}} = \exp\left(-\sum_{\vec{k} \in \mathbb{Z}^3} \log(1 + p_{\vec{k}}) \right) = \exp\left(-\sum_{\vec{k} \in \mathbb{Z}^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} p_{\vec{k}}^n \right)$$

$$P_1 = \sum_{\vec{k} \in \mathbb{Z}^3} \frac{p_{\vec{k}}}{1 + p_{\vec{k}}} \prod_{\vec{l} \in \mathbb{Z}^3} \frac{1}{1 + p_{\vec{l}}} = \sum_{\vec{k} \in \mathbb{Z}^3} \sum_{n=1}^{\infty} (-1)^{n+1} p_{\vec{k}}^n \exp\left(-\sum_{\vec{k} \in \mathbb{Z}^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} p_{\vec{k}}^n \right),$$

etc. . .

Remark 2.1. It is a well-known fact that the photon emission by a classical electric body follows a stochastic Poisson process (see Itzykson, 1980), however from the expression of P_1 we deduce that pair production is not a stochastic Poisson process.

2.2. The Schwinger Formula for Scalar Particles

We consider the case $\vec{f}(t) = (0, 0, \chi(t))$, where

$$\chi(t) = \begin{cases} -cET & \text{if} \quad t < -T \\ cEt & \text{if} \quad -T < t < T \\ cET & \text{if} \quad t > T, \end{cases}$$

with $T \gg 1$. We suppose for example $eE > 0$ (the case $eE < 0$ is analogous).

The Schwinger formula gives the probability that the vacuum state remains unchanged. Then, using the notation

$$N \equiv \frac{2TL^3E^2\alpha}{8\pi^3\hbar}, \qquad S \equiv \frac{\pi m^2 c^4}{\hbar c e E},$$

the Schwinger formula for spinless particles is (Schwinger, 1951)

$$|(0|0\rangle|^2 = \exp\left(-N \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \exp(-nS) \right),$$

where $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant, $|0\rangle \equiv \prod_{\vec{k} \in \mathbb{Z}^3} |0_{\vec{k}}\rangle$ is the vacuum "in"-state, and $|0) \equiv \prod_{\vec{k} \in \mathbb{Z}^3} |0_{\vec{k}})$ is the vacuum "out"-state.

To deduce this formula, we compute the penetration factor, using the relativistic tunneling effect, i.e., using the "formal" W.K.B. method in the complex plane. The result, explained in detail in Haro (in press), is

$$p_{\vec{k}} = \begin{cases} \exp\left(-\frac{1}{\hbar} Im \int_\gamma \sqrt{\left(c^2 p_\perp^2 + m^2c^4 + c^2\left(\frac{2\pi\hbar k_3}{L} + eEz\right)^2\right)} dz\right) & \text{if} \quad \left|\frac{2\pi\hbar k_3}{L}\right| < eET \\ 0 & \text{if} \quad \left|\frac{2\pi\hbar k_3}{L}\right| > eET, \end{cases}$$

where $p_\perp = \frac{2\pi\hbar}{L}(k_1, k_2)$, and γ is a simple curve in the complex plane, containing the complex turning points $(\frac{-2\pi\hbar k_3}{L} \pm \sqrt{p_\perp^2 + m^2c^2})/eE$ as interior points. Now, it easy to verify that

$$p_{\vec{k}} = \begin{cases} \exp\left(-\frac{\pi(c^2p_\perp^2 + m^2c^4)}{\hbar ceE}\right) & \text{if} \quad \left|\frac{2\pi\hbar k_3}{L}\right| < eET \\ 0 & \text{if} \quad \left|\frac{2\pi\hbar k_3}{L}\right| > eET. \end{cases} \tag{3}$$

Then, using this penetration factor, the probability that the vacuum state remains unchanged is

$$P_0 = |\langle 0|0\rangle|^2 = \prod_{\vec{k}\in\mathbb{Z}^3} \frac{1}{1+p_{\vec{k}}} = \exp\left(-\sum_{\vec{k}\in\mathbb{Z}^3} \log(1+p_{\vec{k}})\right)$$
$$= \exp\left(-\sum_{\vec{k}\in\mathbb{Z}^3}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} p_{\vec{k}}^n\right) = \exp\left(-N\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\exp(-nS)\right),$$

in agreement with the Schwinger result.

Remark 2.2. The original method to derive the Schwinger's formula is founded in the definition of a formal effective vacuum action, namely W (Greiner *et al.*, 1985; Grib *et al.*, 1994; Itzykson and, Zuber, 1980; Schwinger, 1951). It is possible to define this action if we assume that the external potential vanish when $t \to \pm\infty$, then the effective vacuum action verifies the equation $\langle 0|0\rangle = \exp(\frac{iw}{\hbar})$.

Using the formal solution of the quantum field equation (obtained form the perturbation theory; Itzykson and Zuber, 1980) it is possible to obtain the following expression

$$W = \frac{i\hbar}{2}\int_0^\infty \frac{ds}{s}\exp(-im^2c^2) Tr\left[\exp\left(is\left(\hat{P} - \frac{e}{c}A\right)^2\right) - \exp(is\,\hat{P}^2)\right],$$

where A is the external potential. Therefore, if we take the potential $A = (0, 0, 0, cEt)$ from this formal expression, it is easy to derive the Schwinger formula.

Remark 2.3. For a Sauter potential, we do not obtain exactly the Schwinger result. In fact, for the following Sauter-type potential $\chi(t) = cET\tanh(\frac{t}{T})$, the exact transmission coefficient is (see Grib *et al.*, 1994; Nikishov, 1970)

$$|T_{\vec{k}}|^2 = \frac{\sinh(2\pi\mu_+)\sinh(2\pi\mu_-)}{\sinh^2(\pi(\mu_+ + \mu_-)) + \cos^2(\pi\lambda)},$$

where

$$\mu_\pm = \frac{T}{2\hbar}\sqrt{c^2 p_\perp^2 + m^2c^4 + c^2\left(\frac{2\pi\hbar k_3}{L} \pm eET\right)^2}; \quad \lambda = \frac{1}{2}\sqrt{1 - \left(\frac{2ceET^2}{\hbar}\right)^2}.$$

Then, in the case $T \gg 1$, we have

$$|T_{\vec{k}}|^2 \sim \begin{cases} \left(1 + \exp\left(-\frac{\pi(c^2p_\perp^2 + m^2c^4)}{\hbar ceE}\frac{e^2E^2T^2}{e^2E^2T^2 - \frac{4\pi^2\hbar^2k_3^2}{L^2}}\right)\right)^{-1} & \text{if} \quad \left|\frac{2\pi\hbar k_3}{L}\right| < eET \\ 1 & \text{if} \quad \left|\frac{2\pi\hbar k_3}{L}\right| > eET. \end{cases}$$

Therefore, for this transmission coefficient, a simple calculation gives the following result:

$$|\langle 0|0\rangle|^2 = \exp\left(-N\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\exp(-nS)\int_0^1(1-x^2)\exp\left(-\frac{nSx^2}{1-x^2}\right)dx\right).$$

In contrast with the result obtained in Grib *et al*. (1994) and Nikishov (1970), where the authors take $(1 + \exp(-\frac{\pi(c^2p_\perp^2+m^2c^4)}{\hbar ceE}))^{-1}$ as the transmission coefficient, and make the replacement $\int dp_3 \to eET$, in order to obtain Schwinger's result.

From (3), it is easy to verify that in this case the generating function $g(x)$ has the following form:

$$g(x) = \exp\left(N\sum_{n=1}^{\infty}\frac{1}{n^2}\left((-1)^{n+1} + (x-1)^n\right)\exp(-nS)\right).$$

If we use the formula (2) and the Taylor's formula $g(1) = \sum_{n=0}^{\infty}\frac{D^ng(0)}{n!}$, we can compute the average number of produced pairs (Holstein, 1999; Nikishov, 1970),

$$\sum_{n=0}^{\infty} nP_n = \frac{Dg(1)}{g(1)} = N\exp(-S).$$

Now, we compute the probability that a pair is produced, using the formula (2), we obtain

$$P_1 = N\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\exp(-nS)\exp\left(-N\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\exp(-nS)\right)$$

$$= N\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\exp(-nS)P_0.$$

Consequently, from these results, we obtain that the average number of produced pairs per unit volume and per unit time is

$$\frac{E^2\alpha}{8\pi^3\hbar}\ \exp\left(-\frac{\pi m^2c^4}{\hbar ceE}\right),$$

and the "relative" probability that a pair is created per unit volume and time is

$$\frac{E^2\alpha}{8\pi^3\hbar}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\ \exp\left(-n\frac{\pi m^2c^4}{\hbar ceE}\right).$$

Remark 2.4. It is important to note that some authors (Greiner *et al.*, 1985; Itzykson and Zuber, 1980; Popov, 1972; Schwinger, 1951) interpret the quantity

$$\frac{E^2\alpha}{8\pi^3\hbar}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\ \exp\left(-n\frac{\pi m^2c^4}{\hbar ceE}\right)$$

as the probability that a pair is created per unit volume and per unit time.

3. THE SCHWINGER FORMULA FOR FERMIONS

To deduce the Schwinger formula for spin particles, we will use the results obtained in the previous section, in particular, we will use the penetration factor obtained in section 2. Now, let $A_{\vec{k}}$ be the probability that no pairs are produced in the $\vec{k}$-state, and let $B_{\vec{k}}$ be the probability that a pair is produced in the $\vec{k}$-state. Then, using the Pauli Exclusion Principle, we obtain the following equations (Haro, in press; Nikishov, 1970):

$$|A_{\vec{k}}|^2+|B_{\vec{k}}|^2=1,\qquad |B_{\vec{k}}|^2=p_{\vec{k}}.$$

Therefore, the probability that the vacuum state remains unchanged is

$$\begin{aligned}|\langle 0|0\rangle|^2 &= \prod_{\vec{k}\in\mathbb{Z}^3}(1-p_{\vec{k}})^2=\exp\left(2\sum_{\vec{k}\in\mathbb{Z}^3}\log(1-p_{\vec{k}})\right)\\ &=\exp\left(-2\sum_{\vec{k}\in\mathbb{Z}^3}\sum_{n=1}^{\infty}\frac{1}{n}p_{\vec{k}}^{n}\right)=\exp\left(-2N\sum_{n=1}^{\infty}\frac{1}{n^2}\ \exp(-nS)\right).\end{aligned}$$

Now we compute P_1. In this case we obtain

$$\begin{aligned}P_1 &= 2\sum_{\vec{k}\in\mathbb{Z}^3}\frac{p_{\vec{k}}}{1-p_{\vec{k}}}\prod_{\vec{l}\in\mathbb{Z}^3}(1-p_{\vec{l}})^2\\ &=2N\sum_{n=1}^{\infty}\frac{1}{n}\ \exp(-nS)\ \exp\left(-2N\sum_{n=1}^{\infty}\frac{1}{n^2}\ \exp(-nS)\right).\end{aligned}$$

To obtain P_n, we use the generating function,

$$g(x) = \exp\left(-2N\sum_{n=1}^{\infty}\frac{1}{n^2}(1-(x+1)^n)\ \exp(-nS)\right),$$

then, the final result is

$$P_n = \frac{1}{n!}D^n g(0)g(-1).$$

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